PRELIMINARIES OF MATHEMATICAL MODELLING AND DYNAMICAL SYSTEMS

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Definition of Modeling:

Lexical meaning of "**model**" is to represent or demonstrate something. So we can say that **Modelling** is a course of representation or demonstration. Here in a model we basically represent the nature or behaviour of a system and the activities involved in it.

Categories of Modelling:

i) Drawing
ii) Physical structure
iii) Verbal arguments
iv) Mathematical formulation

Definition of Mathematical Modelling:

Mathematical modelling is a procedure of representation or demonstration of a system (physical/biological/chemical/technological/linguistic/social or behavioural) and the activities involved in it with the help of mathematical formulation.

Importance of Mathematical Modelling:

i) Simulation on models rather experiments on actual system and thus economic way for measurements.
ii) Control on parameters.
iii) In favourable situations allows us to predict the future nature of a system.

Factors of Mathematical Modelling:

Scope –Is the model affluent to describe the scope of the problem?

Complexity - Is the model simple or complex in terms of parameters and mathematical formulation involved?

Computations–What is the computation cost i.e. length and time involved in solving the model? Is it reliable?

Accuracy–Does the model produce accurate or close to accurate outputs?

Stability-Is the model stable?

Sensitivity–Does the model misbehave with given perturbations?

Methods of Mathematical Modelling:

Any real world system can be modelled by any of the following three basic methods:

i) **Forward Modeling:** It is a process of solving for outputs with known/assumed inputs. Forward modelling is performed when one uses her/his model to predict what would be observed and uses a comparison of these predictions to the data to infer model parameters.

ii) **Inverse Modeling:** It is a method for solving for inputs and outputs with known states and parameters. Inverse modelling is performed when one uses features of the given data to estimate a set of underlying parameters of the concerned physical model.

iii) **Mixed Modeling:** It is a mixture of forward and inverse modeling.

Deterministic and Stochastic Models:

If every variable state involved in a system can be uniquely and exactly determined by parameters in the model it is coined as a **deterministic model**.

Example:

Area of a triangle can be determined uniquely and exactly by the length of the three sides of that triangle.

If any one of the variable state shows random nature rather than being unique and exact then the corresponding model is called a **stochastic model**.

Example:

Prediction of spread of COVID-19 in next month.

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Dynamical System:

A **dynamical system** is a system which changes with time. Mathematically, if a system can be described by means of interaction of a finite number of variables all of which change with time and if further this change in each variable with respect to time can be described by means of certain function involving these variables where time can be present either explicitly or implicitly is said to be a **dynamical system**. The variables describing a dynamical system are called **state variables**.

Examples:

Motion of a swinging simple pendulum, population growth, spread of a contagious disease, dynamics of share market, motions of celestial bodies etc.

N.B.: The idea of dynamical systems can be extended over space also.

Dimension of a Dynamical System:

The number of state variables involved in a dynamical system is said to be the **dimension** of that dynamical system.

Categorization of Dynamical Systems:

If time is implicitly present in the governing equation (s) of a dynamical system then that dynamical system is said to be an **autonomous dynamical system**.

If time is explicitly present at least once in the governing equation (s) of a dynamical system then that dynamical system is said to be a **non-autonomous dynamical system**.

If all the state variables involved in a dynamical system are discrete in nature then the dynamical system is said to be a **discrete dynamical system** or a **map** or a **cascade**.

If all the state variables involved in a dynamical system are continuous in nature then that dynamical system is said to be a **continuous dynamical system** or a **flow**.

A dynamical system is said to be a **conservative dynamical system** if the total energy is preserved.

A dynamical system is said to be a **dissipative dynamical system** if it exchanges and disperses energy.

Phase Space of a Dynamical System:

For a dynamical system the **phase space** is the space in which all possible states of the system are represented such that each unique point in that space corresponds to one possible state of the system.

Examples:

I) Example of a one dimensional autonomous map:

$$x_{t+1} = x_t + x_t^2$$ [General form: $x_{t+1} = x_t + f(x_t)$]

II) Example of a one dimensional non-autonomous map:

$$x_{t+1} = x_t + (x_t^3 - 1) + e^t$$ [General form: $x_{t+1} = x_t + f(t, x_t)$]

III) Example of a two dimensional autonomous map:

$$x_{t+1} = x_t + x_t^2 - 1$$
$$y_{t+1} = y_t + x_t y_t - 1$$

[General form: $x_{t+1} = x_t + f(x_t, y_t)$, $y_{t+1} = y_t + g(x_t, y_t)$]

IV) Example of a two dimensional non-autonomous map:

$$x_{t+1} = x_t + t x_t^3 - 1$$
$$y_{t+1} = y_t + x_t y_t + 1$$

[General form: $x_{t+1} = x_t + f(t, x_t, y_t)$, $y_{t+1} = y_t + g(t, x_t, y_t)$]

V) Example of a one dimensional autonomous flow:

$$\frac{dx}{dt} = x + 1$$ [General form: $\frac{dx}{dt} = f(x)$]

VI) Example of a one dimensional non-autonomous flow:

$$\frac{dx}{dt} = x - 1 + e^t$$

[General form: $\frac{dx}{dt} = f(x,t)$]

VII) Example of a two dimensional autonomous flow:

$$\frac{dx}{dt} = x + y + 2$$

$$\frac{dy}{dt} = xy - 1$$

[General form: $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$]

VIII) Example of a two dimensional non-autonomous flow:

$$\frac{dx}{dt} = x + y + t$$

$$\frac{dy}{dt} = xy - 1$$

[General form: $\frac{dx}{dt} = f(x,y,t)$, $\frac{dy}{dt} = g(x,y,t)$]

We can extend the above ideas for three or higher dimensional maps or flows.

N.B.: In discrete dynamical systems x_t represents the magnitude of x in time t and as derivative does not exist in discrete domain the rate of change of x at t can be equivalently expressed as $\frac{x_{t+1}-x_t}{(t+1)-t} = x_{t+1}-x_t$.

As ordinary differential equation plays its role only in continuous dynamical systems or flows we will confine our analysis within the domain of continuous case. Also we will restrict ourselves in autonomous systems only.

Equilibrium Point of a flow:

One dimension: A point $x=x^* \in D \subseteq R$ is said to be an equilibrium point of a one dimensional flow given by $\frac{dx}{dt} = f(x)$; $x \in D \subseteq \mathbb{R}$ if $\left.\frac{dx}{dt}\right|_{x=x^*} = f(x^*) = 0$.

Two dimension: A point $(x^*, y^*) \in D_2 \subseteq \mathbb{R}^2$ is said to be an equilibrium point of a two dimensional flow given by

$$\left.\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}\right\} \quad (x,y) \in D_2 \subseteq \mathbb{R}^2$$

if

$$\left.\begin{aligned}\frac{dx}{dt}\Big|_{(x^*,y^*)} &= f(x^*,y^*) = 0\\ \frac{dy}{dt}\Big|_{(x^*,y^*)} &= g(x^*,y^*) = 0\end{aligned}\right\}$$

Physically, at an equilibrium point of a flow the flow becomes stationary.

Examples:

I) Given one dimensional flow:

$$\frac{dx}{dt} = 2x - 1 \; ; \quad x \in \mathbb{R}$$

For its equilibrium point we must have

$$\frac{dx}{dt} = 0$$

i.e. $2x - 1 = 0$

or $x = \frac{1}{2}$.

So, $x = \frac{1}{2}$ is its only equilibrium point.

II) Given two dimensional flow:

$$\left.\begin{aligned}\frac{dx}{dt} &= x+y-2\\ \frac{dy}{dt} &= xy-1\end{aligned}\right\} \quad x, y \in \mathbb{R}$$

For its equilibrium point we must have

$$\left.\begin{aligned}\frac{dx}{dt}&=0\\ \frac{dy}{dt}&=0\end{aligned}\right\} \quad \text{i.e.} \quad \begin{aligned}x+y-2&=0\\ xy-1&=0\end{aligned}$$

or, $x=1, y=1$

So, $(1,1)$ is the only equilibrium point of this flow.

There exist certain dynamical systems for which there is no equilibrium point. For example in the one dimensional flow $\frac{dx}{dt} = e^x$; $x \in \mathbb{R}$

$\frac{dx}{dt}$ can never be zero as e^x can never be zero for any $x \in \mathbb{R}$. Hence this flow has no equilibrium point.

Example of Phase Space:

We consider a two dimensional dynamical system as follows:

$$\frac{dx}{dt} = x+1$$

$$\frac{dy}{dt} = xy - x \; ; \; x, y \in \mathbb{R}$$

To find its phase space we take

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{xy-x}{x+1} = \frac{x(y-1)}{x+1}$$

or, $\int \frac{dy}{y-1} = \int \frac{x}{x+1} dx$

$\Rightarrow \ln(y-1) = x - \ln(x+1) + C$

where C is a constant of integration

$\Rightarrow \ln\{(x+1)(y-1)\} = x + C$

$\Rightarrow (x+1)(y-1) = e^{x+C} = ke^x \; [k = e^C]$

$\Rightarrow y - 1 = \frac{ke^x}{x+1} \Rightarrow \boxed{y = 1 + \frac{ke^x}{x+1}}$

The above relation between x and y describes the phase space of the present dynamical system i.e. any pair of (x, y) satisfying above relation lies within this phase space and denotes a unique state of this dynamical system.

Analysis of stability of an equilibrium point of a one dimensional autonomous flow

Let, $\frac{dx}{dt} = f(x)$; $x \in D \subseteq \mathbb{R}$ (1)

be a given one dimensional flow. and let $x = x^* \in D \subseteq \mathbb{R}$ be an equilibrium point of this flow. Then we must have,

$$\left.\frac{dx}{dt}\right|_{x=x^*} = f(x^*) = 0 \qquad (2)$$

We consider a very small amount of perturbation 'Δx' about the equilibrium point $x = x^*$. So near the vicinity of this equilibrium point we have

$$x = x^* + \Delta x \qquad (3)$$

Using (3) in (1) we have,

$$\frac{d}{dt}(x^* + \Delta x) = f(x^* + \Delta x)$$

$$\frac{d}{dt}(\Delta x) = f(x^* + \Delta x)$$

$$= f(x^*) + \Delta x f'(x^*) + \frac{(\Delta x)^2}{2!} f''(x^*)$$

[Using Taylor's series expansion]

i.e. $$\frac{d}{dt}(\Delta x) = \Delta x f'(x^*) + \frac{(\Delta x)^2}{2!} f''(x^*) + \dots \quad (4)$$

[Using (4)]

If 'Δx' is sufficiently small so that we can neglect $(\Delta x)^2$ and other higher powers of Δx then we can have from (4)

$$\frac{d}{dt}(\Delta x) = \Delta x f'(x^*)$$

or $$\frac{d(\Delta x)}{\Delta x} = f'(x^*)\, dt$$

Integrating we get

$$\Delta x = k e^{f'(x^*) t} \quad (5)$$

where 'k' is a constant of integration.

Now, at $t=0$ we assume $\Delta x = \Delta x|_{t=0}$.

So, $$\Delta x|_{t=0} = k. \quad (6)$$

Using (6) in (5) we get

$$\Delta x = \Delta x|_{t=0}\, e^{f'(x^*) t} \quad (7)$$

Case I: $f'(x^*) > 0$:

As $t \to \infty$, $\Delta x \to \infty$ or $-\infty$ according as $\Delta x|_{t=0} > 0$ or < 0 respectively.

In this case, the small perturbation created about the equilibrium point increases with time and eventually goes away from the equilibrium point. This situation represents instability and the corresponding equilibrium point $x = x^*$ is said to be an unstable equilibrium point.

Case II: $f'(x^*) < 0$:

As $t \to \infty$, $\Delta x \to 0$.

In this case, the small perturbation created about the equilibrium point decreases with time and thus tends to return back to the equilibrium point. This situation represents stability and the corresponding equilibrium point $x = x^*$ is said to be an stable equilibrium point.

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Case III: $f'(x^*) = 0$:

We have, $\Delta x = \Delta x|_{t=0} \quad \forall\, t.$

So, here we fail to determine whether the equilibrium point is stable or unstable. Further investigation is required in this case.

Examples:

1. Given one dimensional flow:

$$\frac{dx}{dt} = x^2 - 3x + 2 \quad ; \quad x \in \mathbb{R}.$$

Find its equilibrium point(s) and discuss about the stability.

Ans: Given one dimensional flow:

$$\frac{dx}{dt} = x^2 - 3x + 2 \quad ; \quad x \in \mathbb{R}$$

For its equilibrium point we must have,

$$\frac{dx}{dt} = 0 \quad \text{i.e.} \quad x^2 - 3x + 2 = 0 \quad \text{or} \quad x = 1, 2.$$

So, the given flow has two equilibrium points viz. $x = 1$ and $x = 2$.

We consider $f(x) = x^2 - 3x + 2$.

Hence $f'(x) = 2x - 3$.

Now $f'(1) = 2 \times 1 - 3 = -1 < 0$

So, $x = 1$ is a stable equilibrium point.

Again, $f'(2) = 2 \times 2 - 3 = 1 > 0$.

So, $x = 2$ is an unstable equilibrium point.

2. Given one dimensional flow:

$$\frac{dx}{dt} = 2x^2 \; ; \; x \in \mathbb{R}$$

Find its equilibrium point(s) and discuss about the stability.

Ans:

Given one dimensional flow:

$$\frac{dx}{dt} = 2x^2 \; ; \; x \in \mathbb{R}.$$

For its equilibrium point we must have

$$\frac{dx}{dt} = 0 \text{ i.e. } 2x^2 = 0 \text{ or } x = 0.$$

So, $x = 0$ is its only equilibrium point.

Now, we take, $f(x) = 2x^2$

So, $f'(x) = 4x$

and $f'(0) = 0$

Hence no conclusion can be drawn about the stability of the equilibrium point $x=0$ from the above. Now, if we consider 'Δx' as the small perturbation about the equilibrium point $x=0$ we then have near the vicinity of this equilibrium point $x = 0 + \Delta x$ i.e. $x = \Delta x$.

Then we get,

$$\frac{d}{dt}(\Delta x) = f(\Delta x)$$

$$= f(0) + \Delta x f'(0) + \frac{(\Delta x)^2}{2!} f''(0) + \frac{(\Delta x)^3}{3!} f'''(0) + \ldots,$$

Now, $f(x) = 2x^2$

$f'(x) = 4x$

$f''(x) = 4$ and $f^{n}(x) = 0 \quad \forall\, n \geq 3.$

So, $\frac{d}{dt}(\Delta x) = 0 + \Delta x \cdot 0 + \frac{(\Delta x)^2}{2!} \times 4 + 0$

$= 2(\Delta x)^2$

or, $\frac{d(\Delta x)}{(\Delta x)^2} = 2\,dt$

Integrating we get,

$-\frac{1}{\Delta x} = 2t + k'$ where k' is a constant of integration.

or, $\Delta x = -\frac{1}{2t + k'}$

Now, as $t \to \infty$, $\Delta x \to 0$.

So, from the above analysis we get that $x = 0$ is a stable equilibrium point of the given flow.

One periodic fixed point of a Map and Analysis of stability of one-periodic fixed point of a one dimensional autonomous map

Let $x_{t+1} = x_t + f(x_t)$; $x \in D \subseteq R$ → (1)
be a given one dimensional map. For one periodic fixed point we must have,

$$x_{t+1} = x_t \Rightarrow f(x_t) = 0. \quad \rightarrow (2)$$

Now, we take, $x + f(x) = g(x)$ → (3)

Then from (1) using (3) we get,

$$x_{t+1} = g(x_t) \rightarrow (4).$$

Now, if we take, $x_t = x^*$ as a one-periodic fixed point then above system (4) acts as a fixed point iteration which will converge to x^* if the one-periodic fixed point $x_t = x^*$ is stable. Now, for convergence of (4) we must have $|g'(x^*)| < 1$ i.e. $-1 < g'(x^*) < 1$

or, $-1 < 1 + f'(x^*) < 1 \Rightarrow -2 < f'(x^*) < 0.$

So, the one-periodic fixed point $x_t = x^*$ will be stable if we have $-2 < f'(x^*) < 0.$

Examples:

1. Find the one-periodic fixed point(s) and discuss about the stability for the one dimensional map:

$$x_{t+1} = x_t + x_t^3 - 8, \quad x \in \mathbb{R}.$$

Ans: Given one dimensional map:

$$x_{t+1} = x_t + x_t^3 - 8 \; ; \; x \in \mathbb{R}$$

for one periodic fixed point we must have,

$$x_{t+1} = x_t$$

So, $x_t = x_t + x_t^3 - 8 \Rightarrow x_t^3 = 8 \Rightarrow x_t = 2$.

So, one periodic fixed point, $x_t = 2$.

Now, here, $f(x) = x^3 - 8$. [Standard one dimensional map: $x_{t+1} = x_t + f(x_t)$]

$\therefore f'(x) = 3x^2$.

Now, $f'(2) = 3 \times 2^2 = 12 > 0$.

So, $x_t = 2$ is an unstable one-periodic fixed point.

2. Find the one-periodic fixed point(s) and discuss about the stability for the one-dimensional map:

$$x_{t+1} = x_t + 2x_t^2 - 8 \quad ; \quad x \in \mathbb{R}$$

Ans: Given one dimensional map:

$$x_{t+1} = x_t + 2x_t^2 - 8 \quad ; \quad x \in \mathbb{R}$$

For one periodic fixed point we must have

$$x_{t+1} = x_t$$

So, $x_t = x_t + 2x_t^2 - 8$

So, one periodic fixed points: $\Rightarrow 2x_t^2 = 8 \Rightarrow x_t^2 = 4 \Rightarrow x_t = \pm 2$.
$x_t = \pm 2$

Here, $f(x) = 2x^2 - 8$, [Standard one dimensional map; $x_{t+1} = x_t + f(x_t)$]

$\therefore f'(x) = 4x$

Here, $f'(2) = 8 > 0$

and $f'(-2) = 4(-2) = -8 < -2$.

So, both the one periodic fixed points ± 2 are unstable one-periodic fixed points. [for stability we must have $-2 < f'(x^*) < 0$]

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3. Find the one periodic fixed point(s) and discuss about the stability for the following one dimensional map:

$$x_{t+1} = x_t^2 - x_t + 1 \quad ; \quad x \in \mathbb{R}$$

Ans:

Given one dimensional map:

$$x_{t+1} = x_t^2 - x_t + 1 \quad ; \quad x \in \mathbb{R}$$

$\Rightarrow x_{t+1} = x_t + (x_t^2 - 2x_t + 1) \Rightarrow x_{t+1} = x_t + (x_t - 1)^2$

for one-periodic fixed point we must have,

$$x_{t+1} = x_t.$$

So, $x_t = x_t + (x_t - 1)^2$

$\Rightarrow (x_t - 1)^2 = 0 \Rightarrow x_t = 1.$ So, only one periodic fixed point: $x_t = 1$.

Here, $f(x) = (x-1)^2$

$\therefore f'(x) = 2(x-1).$

$\therefore f'(1) = 2(1-1) = 0.$

So, $x_t = 1$ is an unstable one periodic fixed point. [for stability: $-2 < f'(x^*) < 0$]

4. Find the one-periodic fixed points and discuss about the stability for the following one dimensional map:

$$x_{t+1} = x_t + mx_t^2 - 4 \quad (m>0); \quad x \in \mathbb{R}.$$

Ans: Given one dimensional map:

$$x_{t+1} = x_t + mx_t^2 - 4 \quad (m>0); \quad x \in \mathbb{R}.$$

For one periodic fixed point we must have

$$x_{t+1} = x_t$$

So, $x_t = x_t + mx_t^2 - 4$

$$\Rightarrow mx_t^2 - 4 = 0 \Rightarrow x_t = \pm \frac{2}{\sqrt{m}} \quad (m>0).$$

So, the one-periodic fixed points:

$$x_t = \pm \frac{2}{\sqrt{m}}.$$

Here, $f(x) = mx^2 - 4$.

$\therefore f'(x) = 2mx.$

$\therefore f'\left(\frac{2}{\sqrt{m}}\right) = 2m \times \frac{2}{\sqrt{m}} = 4\sqrt{m} > 0.$

So, $x_t = \frac{2}{\sqrt{m}}$ is an unstable one periodic fixed point.

Again, $f'(-\frac{2}{\sqrt{m}}) = 2m \times (-\frac{2}{\sqrt{m}}) = -4\sqrt{m}$.

For stability we must have, $-2 < f'(-\frac{2}{\sqrt{m}}) < 0$

$\Rightarrow -2 < -4\sqrt{m} < 0$

$\Rightarrow 0 < 4\sqrt{m} < 2$

$\Rightarrow \sqrt{m} < \frac{1}{2}$

$\Rightarrow m < \frac{1}{4}$.

$\therefore$ for $0 < m < \frac{1}{4}$ the one-periodic fixed point $(-\frac{2}{\sqrt{m}})$ is stable. for $m \geq \frac{1}{4}$, it is unstable.

5. Find the one-periodic fixed points and discuss about the stability for the following one dimensional map:

$$x_{t+1} = a x_t (1 - x_t) \; ; \; x \in \mathbb{R}$$

Ans: Given one dimensional map:

$$x_{t+1} = a x_t (1 - x_t) \; ; \; x \in \mathbb{R}$$

or, $x_{t+1} = x_t + (a-1) x_t - a x_t^2$

Now, for one periodic fixed point we must have, $x_{t+1} = x_t$.

So, $x_t = x_t + (a-1)x_t - ax_t^2$

$\Rightarrow (a-1)x_t = ax_t^2$.

$\Rightarrow x_t = 0, \frac{a-1}{a}$.

Now, $f(x) = (a-1)x - ax^2$

$\therefore f'(x) = a-1-2ax$

$\therefore f'(0) = a-1$

So, for stability we must have,

$-2 < f'(0) < 0 \Rightarrow -2 < a-1 < 0$

$\Rightarrow -1 < a < 1$.

So, if $-1 < a < 1$ then $x_t = 0$ is a stable one-periodic fixed point. Otherwise for $a \geq 1$ or for $a \leq -1$ it is unstable.

Again, $f'\left(\frac{a-1}{a}\right) = a-1-2a\left(\frac{a-1}{a}\right)$

$= a-1-2(a-1) = -(a-1)$

$= 1-a$

For stability we must have,

$-2 < f'\left(\frac{a-1}{a}\right) < 0 \Rightarrow -2 < 1-a < 0$

$\Rightarrow 0 < a-1 < 2 \Rightarrow 1 < a < 3$.

So, if $1 < a < 3$ then $x_t = \frac{a-1}{a}$ is a stable one-periodic fixed point. Otherwise for $a \geq 3$ or $a \leq 1$ it is unstable.

Analysis of stability of an equilibrium point of a two dimensional autonomous flow

Let,

$$\left.\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}\right\} \quad (x,y) \in D_2 \subseteq \mathbb{R} \qquad (1)$$

be a given two dimensional flow and let $(x^*, y^*) \in D_2 \subseteq \mathbb{R}$ be an equilibrium point of this flow. Then we must have

$$\left.\begin{aligned}\left.\frac{dx}{dt}\right|_{(x^*,y^*)} &= f(x^*,y^*) = 0\\ \left.\frac{dy}{dt}\right|_{(x^*,y^*)} &= g(x^*,y^*) = 0\end{aligned}\right\} \qquad (2)$$

We consider a very small amount of perturbation given by $(\Delta x, \Delta y)$ about the equilibrium point (x^*, y^*). So, near the vicinity of the equilibrium point (x^*, y^*) we have

$$\left.\begin{aligned} x &= x^* + \Delta x \\ y &= y^* + \Delta y \end{aligned}\right\} \qquad (3)$$

Using (3) in (1) we get,

$$\frac{d}{dt}(x^* + \Delta x) = f(x^* + \Delta x, y^* + \Delta y)$$

$$\frac{d}{dt}(y^* + \Delta y) = g(x^* + \Delta x, y^* + \Delta y)$$

or, $$\frac{d}{dt}(\Delta x) = f(x^*, y^*) + \Delta x \cdot \frac{\partial f}{\partial x}(x^*, y^*)$$

$$+ \frac{(\Delta x)^2}{2!} \frac{\partial^2 f}{\partial x^2}(x^*, y^*) + \ldots$$

$$+ \Delta y \cdot \frac{\partial f}{\partial y}(x^*, y^*) + \frac{(\Delta y)^2}{2!} \frac{\partial^2 f}{\partial y^2}(x^*, y^*) + \ldots$$

$$\frac{d}{dt}(\Delta y) = g(x^*, y^*) + \Delta x \cdot \frac{\partial g}{\partial x}(x^*, y^*)$$

$$+ \frac{(\Delta x)^2}{2!} \frac{\partial^2 g}{\partial x^2}(x^*, y^*) + \ldots$$

$$+ \Delta y \cdot \frac{\partial g}{\partial y}(x^*, y^*) + \frac{(\Delta y)^2}{2!} \frac{\partial^2 g}{\partial y^2}(x^*, y^*) + \ldots$$

This gives,

$$\frac{d}{dt}(\Delta x) = \frac{\partial f}{\partial x}(x^*, y^*).\Delta x + \frac{\partial f}{\partial y}(x^*, y^*).\Delta y$$

$$\frac{d}{dt}(\Delta y) = \frac{\partial g}{\partial x}(x^*, y^*)\Delta x + \frac{\partial g}{\partial y}(x^*, y^*)\Delta y \quad \} (4)$$

Using (2) and considering Δx and Δy sufficiently small so that their squares and other higher powers can be neglected.

(4) can be equivalently written as

$$\frac{d}{dt}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}_{(x^*, y^*)} \begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} \quad (5)$$

If we take, $\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} \approx X$ and $\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}$

which represents the Jacobian of the system as J we have

from (5)

$$\frac{d\underset{\sim}{X}}{dt} = J_{(x^*,y^*)}\,\underset{\sim}{X} \qquad (6)$$

We take a trial solution of (6) as

$$\underset{\sim}{X} = \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda t} \qquad (7)$$

Then we have, $\frac{d\underset{\sim}{X}}{dt} = \lambda \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda t}$ (8)

Using (7) and (8) in (6) we get

$$\lambda \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda t} = J_{(x^*,y^*)} \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda t}$$

$$\lambda \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda t} = e^{\lambda t} J_{(x^*,y^*)} \begin{bmatrix} c \\ d \end{bmatrix}$$

or, $J_{(x^*,y^*)} \begin{bmatrix} c \\ d \end{bmatrix} = \lambda \begin{bmatrix} c \\ d \end{bmatrix}$ (9)

From (9) it is clear that λ is an eigen value of $J(x^*, y^*)$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is its corresponding eigen vector. The corresponding characteristic equation is

$$\det(J - \lambda I) = 0$$

i.e.
$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix}_{(x^*, y^*)} = 0$$

or, $\left(\frac{\partial f}{\partial x} - \lambda\right)\left(\frac{\partial g}{\partial y} - \lambda\right) - \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x} = 0$

or, $\lambda^2 - \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)_{(x^*, y^*)} \lambda + \left(\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x}\right)_{(x^*, y^*)} = 0$ (10)

The above is a quadratic equation of λ. We can arrive at the solution for λ for different cases as given below.

Case I: Roots are real and unequal: (say, λ_1 and λ_2)

[The corresponding equilibrium point is said to be a node]

Here we have

$$\Delta x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$
$$\Delta y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Subcase Ia: $\lambda_1 > 0,\ \lambda_2 > 0.$

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$. (If $C_1, C_2 > 0$)

or, $\Delta x \to -\infty$, $\Delta y \to -\infty$ (If $C_1, C_2 < 0$).

Hence, the equilibrium point is an unstable node.

Subcase Ib: $\lambda_1 < 0,\ \lambda_2 < 0$:

As $t \to \infty$, $\Delta x \to 0$, $\Delta y \to 0$.

Hence, the equilibrium point is a stable node.

Subcase Ic: $\lambda_1 > 0,\ \lambda_2 < 0$, or, $\lambda_1 < 0,\ \lambda_2 > 0$:

As $t \to \infty$, one component tends to infinity and the other component drags it to zero. In this situation the corresponding equilibrium point is said to be a saddle node.

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Subcase Id: $\lambda_1 = 0,\ \lambda_2 > 0$ or $\lambda_1 > 0,\ \lambda_2 = 0$:

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$ (if $c_1, c_2 > 0$) or $\Delta x \to -\infty$, $\Delta y \to -\infty$ (if $c_1, c_2 < 0$)

Hence the equilibrium point is said to be an unstable node.

Subcase Ie: $\lambda_1 = 0,\ \lambda_2 < 0$ or $\lambda_1 < 0,\ \lambda_2 = 0$:

As $t \to \infty$, $\Delta x \to c_1$, $\Delta y \to c_1$ or $\Delta x \to c_2$, $\Delta y \to c_2$.

Here, we call the equilibrium point as a pseudo-stable node.

Case II: [Roots are real and equal (say λ^* and λ^*)

[Here also the corresponding equilibrium point is said to be a node]

Here we have

$$\left.\begin{aligned} \Delta x &= (c_1' + c_2' t)\, e^{\lambda^* t} \\ \Delta y &= (c_1' + c_2' t)\, e^{\lambda^* t} \end{aligned}\right\}$$

Subcase IIa': $\lambda^* > 0$;

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$ (if $C_1, C_2 > 0$) or

$\Delta x \to -\infty$, $\Delta y \to -\infty$ (if $C_1, C_2 < 0$).

Here the equilibrium point is an unstable node.

Subcase IIb': $\lambda^* < 0$;

As $t \to \infty$; $\Delta x \to 0$, $\Delta y \to 0$.

Here the equilibrium point is a stable node.

Subcase IIc': $\lambda^* = 0$;

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$ (if $C_1, C_2 > 0$)

or, $\Delta x \to -\infty$, $\Delta y \to -\infty$ (if $C_1, C_2 < 0$).

Here, the equilibrium point is an unstable node.

Case II': Roots are complex conjugate numbers (say $-\alpha \pm i\beta$)

[The corresponding equilibrium point is said to be a focus if $\alpha \neq 0$ and centre if $\alpha = 0$]

Here we have

$$\Delta x = \left[C_1'' \cos(\beta t) + C_2'' \sin(\beta t)\right] e^{\alpha t}$$

$$\Delta y = \left[C_1'' \cos(\beta t) + C_2'' \sin(\beta t)\right] e^{\alpha t}$$

Subcase III a': $\alpha > 0$;

As $t \to \infty$, $|\Delta x| \to \infty$, $|\Delta y| \to \infty$.

Hence the equilibrium point is an unstable focus.

Subcase III b': $\alpha < 0$;

As $t \to \infty$, $\Delta x \to 0$, $\Delta y \to 0$.

Hence the equilibrium point is a stable focus.

Subcase III c': $\alpha = 0$;

Here, as t increases Δx and Δy oscillates between two finite values. Here the equilibrium point is said to be a centre.

Example:

Given two dimensional flow:

$$\left.\begin{aligned}\frac{dx}{dt} &= x(4-x-y)\\ \frac{dy}{dt} &= y(15-5x-3y)\end{aligned}\right\} \quad x, y \in \mathbb{R}$$

Find the equilibrium point(s) and discuss about the stability.

Ans: Given two dimensional flow:

$$\left.\begin{aligned}\frac{dx}{dt} &= x(4-x-y)\\ \frac{dy}{dt} &= y(15-5x-3y)\end{aligned}\right\} \quad x, y \in \mathbb{R}$$

For its equilibrium point we must have

$$\left.\begin{aligned}\frac{dx}{dt} &= 0\\ \frac{dy}{dt} &= 0\end{aligned}\right\} \text{ i.e. } \left.\begin{aligned}x(4-x-y) &= 0\\ y(15-5x-3y) &= 0\end{aligned}\right\}$$

Option 1: $x = 0, y = 0$.

Hence $(0,0)$ is an equilibrium point.

Option 2: $x=0,\ 15-5x-3y=0$

i.e. $x=0, y=5$.

Hence $(0,5)$ is an equilibrium point.

Option 3: $y=0,\ 4-x-y=0$

i.e. $x=4, y=0$,

Hence $(4,0)$ is an equilibrium point.

Option 4: $4-x-y=0;\ 15-5x-3y=0$

Solving we get $x=\frac{3}{2}, y=\frac{5}{2}$.

Hence, $\left(\frac{3}{2}, \frac{5}{2}\right)$ is an equilibrium point.

Therefore for the given flow we have four equilibrium points viz. $(0,0)$; $(0,5)$; $(4,0)$ and $\left(\frac{3}{2}, \frac{5}{2}\right)$.

We take,

$$f(x,y) = x(4-x-y)$$
$$g(x,y) = y(15-5x-3y)$$

So, $\frac{\partial f}{\partial x} = 4-2x-y$; $\frac{\partial f}{\partial y} = -x$; $\frac{\partial g}{\partial x} = -5y$;
$\frac{\partial g}{\partial y} = 15-5x-6y$.

Therefore general Jacobian of the system

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 4-2x-y & -x \\ -5y & 15-5x-6y \end{bmatrix}$$

Stability Analysis of the equilibrium points:

I $(0,0)$:

Characteristic equation:

$$\det(J-\lambda I)_{(0,0)} = 0$$

i.e. $$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 15-\lambda \end{vmatrix} = 0$$

or $(4-\lambda)(15-\lambda)=0$ i.e. $\lambda = 4, 15$.

As here both the eigen values are positive $(0,0)$ is an unstable node.

II. $(0, 5)$;

Characteristic equation:

$$\det(J - \lambda I)_{(0,5)} = 0$$

i.e.
$$\begin{vmatrix} -1-\lambda & 0 \\ -25 & -15-\lambda \end{vmatrix} = 0$$

i.e. $(-1-\lambda)(-15-\lambda) = 0$ or, $\lambda = -1, -15$.

As here both the eigen values are negative $(0, 5)$ is a stable node.

III. $(4, 0)$;

Characteristic equation:

$$\det(J - \lambda I)_{(4,0)} = 0$$

i.e.
$$\begin{vmatrix} -4-\lambda & -4 \\ 0 & -5-\lambda \end{vmatrix} = 0$$

i.e. $(-4-\lambda)(-5-\lambda) = 0$ or $\lambda = -4, -5$.

As here both the eigen values are negative $(4,0)$ is a stable node.

IV. $\left(\frac{3}{2}, \frac{5}{2}\right)$:

Characteristic equation:

$$\det (J - \lambda I)_{\left(\frac{3}{2}, \frac{5}{2}\right)} = 0$$

i.e. $$\begin{vmatrix} -\frac{3}{2} - \lambda & -\frac{3}{2} \\ -\frac{25}{2} & -\frac{15}{2} - \lambda \end{vmatrix} = 0$$

or $$\left(-\frac{3}{2} - \lambda\right)\left(-\frac{15}{2} - \lambda\right) - \left(-\frac{3}{2}\right)\left(-\frac{25}{2}\right) = 0$$

or $$\lambda^2 + 9\lambda - \frac{15}{2} = 0$$

or, $$\lambda = \frac{-9 \pm \sqrt{9^2 - 4 \times 1 \times \left(-\frac{15}{2}\right)}}{2 \times 1}$$

$$= \frac{-9 \pm \sqrt{101}}{2}$$

Here one root is negative and the other is positive. Hence, $\left(\frac{3}{2}, \frac{5}{2}\right)$ is a saddle node.

Routh Hurwitz Criteria for stability of an equilibrium point for an autonomous flow

For three dimensional flow:

We consider the following three dimensional autonomous flow:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(x,y,z) \\ \frac{dy}{dt} &= f_2(x,y,z) \\ \frac{dz}{dt} &= f_3(x,y,z) \end{aligned}\right\} \quad (1)$$

Let (x^*, y^*, z^*) be an equilibrium point of the above flow. Then we must have

$$\left.\begin{aligned} \frac{dx}{dt}\Big|_{(x^*,y^*,z^*)} &= f_1(x^*,y^*,z^*) = 0 \\ \frac{dy}{dt}\Big|_{(x^*,y^*,z^*)} &= f_2(x^*,y^*,z^*) = 0 \\ \frac{dz}{dt}\Big|_{(x^*,y^*,z^*)} &= f_3(x^*,y^*,z^*) = 0 \end{aligned}\right\} \quad (2)$$

Now, for the equilibrium point (x^*, y^*, z^*) we arrive at the following characteristic equation:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} - \lambda & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} - \lambda & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} - \lambda \end{vmatrix} = 0 \quad (x^*, y^*, z^*)$$

$$\Rightarrow p_0 + p_1\lambda + p_2\lambda^2 + p_3\lambda^3 = 0 \quad \text{(Δ eq)}$$

[p_3 is preferrably 1]

We consider the following matrix:

$$L_3 = \begin{bmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{bmatrix}$$

According to Routh-Hurwitz Criteria the equilibrium point (x^*, y^*, z^*) will be stable if i) $p_0 > 0$; ii) $D_1 = p_1 > 0$;

iii) $D_2 = \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix} = p_1p_2 - p_0p_3 > 0.$

Otherwise it is unstable.

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For four dimensional autonomous flow:

We consider the following four dimensional autonomous flow:

$$\left.\begin{aligned}\frac{dx}{dt} &= f_1(x,y,z,w)\\ \frac{dy}{dt} &= f_2(x,y,z,w)\\ \frac{dz}{dt} &= f_3(x,y,z,w)\\ \frac{dw}{dt} &= f_4(x,y,z,w)\end{aligned}\right\} \longrightarrow (1)$$

Let (x^*, y^*, z^*, w^*) be an equilibrium point of the above flow. Then we must have

$$\left.\begin{aligned}\frac{dx}{dt}\Big|_{(x^*,y^*,z^*,w^*)} &= f_1(x^*,y^*,z^*,w^*) = 0\\ \frac{dy}{dt}\Big|_{(x^*,y^*,z^*,w^*)} &= f_2(x^*,y^*,z^*,w^*) = 0\\ \frac{dz}{dt}\Big|_{(x^*,y^*,z^*,w^*)} &= f_3(x^*,y^*,z^*,w^*) = 0\\ \frac{dw}{dt}\Big|_{(x^*,y^*,z^*,w^*)} &= f_4(x^*,y^*,z^*,w^*) = 0\end{aligned}\right\} (2)$$

~~Here let us assume that~~

Now for the equilibrium point $(x^*, y^*, z^*, \omega^*)$ we arrive at the following characteristic equation:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} - \lambda & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial \omega} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} - \lambda & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial \omega} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} - \lambda & \frac{\partial f_3}{\partial \omega} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial \omega} - \lambda \end{vmatrix}_{(x^*, y^*, z^*, \omega^*)} = 0$$

$$\Rightarrow p_0 + p_1\lambda + p_2\lambda^2 + p_3\lambda^3 + p_4\lambda^4 = 0 \quad \text{(say)}$$

[p_4 is preferrably 1]

We consider the following matrix:

$$L_4 = \begin{bmatrix} p_1 & p_0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & 0 & p_4 \end{bmatrix}$$

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According to Routh-Hurwitz criteria the equilibrium point (x^*, y^*, z^*, w^*) will be stable if i) $\rho_0 > 0$; ii) $D_1 = \rho_1 > 0$; iii) $D_2 = \begin{vmatrix} \rho_1 & \rho_0 \\ \rho_3 & \rho_2 \end{vmatrix} > 0$

$\Rightarrow \rho_1\rho_2 - \rho_0\rho_3 > 0$; iv) $D_3 = \begin{vmatrix} \rho_1 & \rho_0 & 0 \\ \rho_3 & \rho_2 & \rho_1 \\ 0 & \rho_4 & \rho_3 \end{vmatrix} > 0$

$\Rightarrow \rho_1(\rho_2\rho_3 - \rho_1\rho_4) + \rho_0(-\rho_3^2) > 0$

$\Rightarrow \rho_1\rho_2\rho_3 - \rho_1^2\rho_4 - \rho_0\rho_3^2 > 0$

$\Rightarrow \rho_3(\rho_1\rho_2 - \rho_0\rho_3) > \rho_1^2\rho_4$

Otherwise it is unstable.

N.B.! This idea can be generalized for any n-dimensional autonomous flow.

Problems:

1. Find the equilibrium point(s) and discuss about the stability of the following three dimensional flow using Routh-Hurwitz criteria:

$$\frac{dx}{dt} = \alpha x + \beta y$$
$$\frac{dy}{dt} = x - z$$
$$\frac{dz}{dt} = -x + y \quad ; \quad x, y, z \in \mathbb{R}; \ \alpha, \beta \in \mathbb{R}$$

Ans: Given three dimensional flow:

$$\frac{dx}{dt} = \alpha x + \beta y$$

$$\frac{dy}{dt} = x - z$$

$$\frac{dz}{dt} = -x + y$$

Now, for equilibrium point we must have,

$$\frac{dx}{dt} = 0 \Rightarrow \alpha x + \beta y = 0$$

$$\frac{dy}{dt} = 0 \Rightarrow x - z = 0 \Rightarrow x = z$$

$$\frac{dz}{dt} = 0 \Rightarrow -x + y = 0 \Rightarrow x = y.$$

$\therefore x = y = z$ and $\alpha x + \beta y = 0$

$\Rightarrow \alpha x + \beta x = 0$

$\Rightarrow (\alpha + \beta) x = 0$

$\therefore$ either $x = 0$ or $\alpha + \beta = 0$.

If $x = 0$, $y = z = 0$.

And if $\alpha + \beta = 0$ then $x = y = z = k$ (k being any real number)

So, we have the equilibrium point as $(0,0,0)$ (irrespective of α, β) and (k,k,k) where $k \in \mathbb{R}$ where $\alpha + \beta = 0$.

Now, here, $f_1(x,y,z) = \alpha x + \beta y$

$\therefore \frac{\partial f_1}{\partial x} = \alpha \; ; \; \frac{\partial f_1}{\partial y} = \beta \; ; \; \frac{\partial f_1}{\partial z} = 0$

$f_2(x,y,z) = x - z.$

$\therefore \frac{\partial f_2}{\partial x} = 1 \; ; \; \frac{\partial f_2}{\partial y} = 0 \; ; \; \frac{\partial f_2}{\partial z} = -1$

$f_3(x,y,z) = -x + y$

$\therefore \frac{\partial f_3}{\partial x} = -1 \; ; \; \frac{\partial f_3}{\partial y} = 1 \; ; \; \frac{\partial f_3}{\partial z} = 0.$

So, for the equilibrium point $(0,0,0)$ the characteristic equation takes the following form:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} - \lambda & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} - \lambda & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} - \lambda \end{vmatrix} = 0 \quad (0,0,0)$$

$$\Rightarrow \begin{vmatrix} \alpha-\lambda & \beta & 0 \\ 1 & 0-\lambda & -1 \\ -1 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\alpha-\lambda)\{(-\lambda).(-\lambda)-(-1).1\}+\beta\{(-1).(-1)-1.(-\lambda)\} = 0$$

$$\Rightarrow (\alpha-\lambda)(\lambda^2+1)+\beta(1+\lambda)=0$$

$$\Rightarrow \alpha\lambda^2+\alpha-\lambda^3-\lambda+\beta+\beta\lambda=0$$

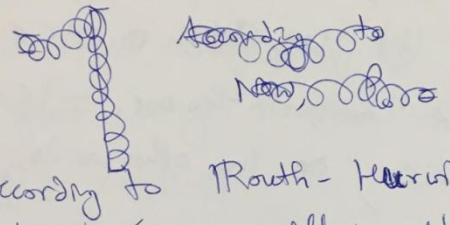
$$\Rightarrow -\lambda^3+\alpha\lambda^2+(\beta-1)\lambda+(\alpha+\beta)=0$$

$$\Rightarrow -(\alpha+\beta)+(1-\beta)\lambda-\alpha\lambda^2+\lambda^3=0.$$

Here, $p_0 = -(\alpha+\beta)$
$p_1 = 1-\beta$
$p_2 = -\alpha$
$p_3 = 1$

We consider the following matrix:

$$L_3 = \begin{bmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{bmatrix}$$



According to Routh-Hurwitz criteria the equilibrium point (0,0,0) will be stable if

i) $p_0 > 0 \Rightarrow -(\alpha+\beta) > 0 \Rightarrow \alpha+\beta < 0 \longrightarrow (1)$

ii) $D_1 = p_1 > 0 \Rightarrow 1-\beta > 0 \Rightarrow \beta < 1 \longrightarrow (2)$

iii) $D_2 = \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix} = p_1 p_2 - p_0 p_3 > 0$

$\Rightarrow (1-\beta)(-\alpha) - \{-(\alpha+\beta)\} \times 1 > 0$

$\Rightarrow -\alpha + \alpha\beta + \alpha + \beta > 0$

$\Rightarrow \beta(\alpha+1) > 0 \longrightarrow (3)$

From (3) we have either $\beta > 0$ and $\alpha > -1$ or, $\beta < 0$, $\alpha < -1$.

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For $\beta<0$, $\alpha<-1$ the inequalities (1) and (2) hold satisfactorily.

For the option $\beta>0$, $\alpha>-1$ we can find (2) gives $0<\beta<1$. Again, $\alpha>-1$.

So, from (1) we get, $-1<\alpha<0$.

Hence we have, in this case, $-1<\alpha<0$; $0<\beta<1$ and $\alpha+\beta<0$, or in other words, $-1<\alpha<0$ and $0<\beta<-\alpha$.

Hence the equilibrium point $(0,0,0)$ is stable if $\alpha<-1$; $\beta<0$ or $-1<\alpha<0$; $0<\beta<-\alpha$.

Otherwise it is unstable.

Again, for the choice of the equilibrium point (k,k,k) where $k\in\mathbb{R}$ and $\alpha+\beta=0$ the characteristics equation remains the same and hence L_3 matrix is the same. Now, according to Routh-Hurwitz criteria we must have,

$P_0 > 0$ i.e. $-(\alpha+\beta) > 0$

But here, $\alpha+\beta=0$. Hence we arrive at a contradiction. So, for $\alpha+\beta=0$, all the equilibrium point (k, k, k); $k \in \mathbb{R}$ are unstable.

2. Find the equilibrium point(s) and discuss about the stability of the following three dimensional flow using Routh–Hurwitz criteria:

$$\frac{dx}{dt} = x+y(1-y)$$

$$\frac{dy}{dt} = x(1-x) - y + z(2-z)$$

$$\frac{dz}{dt} = 2x + y(1-y) \quad ; x, y, z \in \mathbb{R}.$$

Ans: Given three dimensional flow:

$$\frac{dx}{dt} = x+y(1-y)$$

$$\frac{dy}{dt} = x(1-x) - y + z(2-z)$$

$$\frac{dz}{dt} = 2x + y(1-y) \quad ; x, y, z \in \mathbb{R}$$

For equilibrium point we must have

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$$\frac{dx}{dt}=0 \Rightarrow x+y(1-y)=0 \quad \rightarrow (1)$$

$$\frac{dy}{dt}=0 \Rightarrow x(1-x)-y+z(2-z)=0 \quad \rightarrow (2)$$

$$\frac{dz}{dt}=0 \Rightarrow 2x+y(1-y)=0 \quad \rightarrow (3)$$

Now, (3) − (1) gives,

$$x=0.$$

Putting $x=0$, in (1) we get

$$y(1-y)=0 \Rightarrow y=0,1.$$

From (2) for $x=0$, $y=0$, we have,

$$z(2-z)=0 \Rightarrow z=0,2.$$

Hence, $(0,0,0)$ and $(0,0,2)$ are here equilibrium points,

Again, for $x=0$, $y=1$ we have from (2)

$$-1+z(2-z)=0$$

$$\Rightarrow -z^2+2z-1=0$$

$$\Rightarrow (z-1)^2=0 \Rightarrow z=1.$$

$\therefore$ $(0,1,1)$ is another equilibrium point.

Hence we have three equilibrium points viz. $(0,0,0)$; $(0,0,2)$ and $(0,1,1)$.

Now, here $f_1(x,y,z) = x+y(1-y)$

$\therefore \frac{\partial f_1}{\partial x} = 1$; $\frac{\partial f_1}{\partial y} = 1-2y$; $\frac{\partial f_1}{\partial z} = 0$

$f_2(x,y,z) = x(1-x) - y + z(2-z)$

$\frac{\partial f_2}{\partial x} = 1-2x$; $\frac{\partial f_2}{\partial y} = -1$; $\frac{\partial f_2}{\partial z} = 2-2z$

$f_3(x,y,z) = 2x+y(1-y)$

$\frac{\partial f_3}{\partial x} = 2$; $\frac{\partial f_3}{\partial y} = 1-2y$; $\frac{\partial f_3}{\partial z} = 0$.

So, the general characteristic equation is

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} - \lambda & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial x} - \lambda & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} - \lambda \end{vmatrix} = 0$$

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$$\Rightarrow \begin{vmatrix} 1-\lambda & 1-2y & 0 \\ 1-2x & -1-\lambda & 2-2z \\ 2 & 1-2y & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1-2y & 0 \\ 1-2x & -1-\lambda & 2-2z \\ 2 & 1-2y & -\lambda \end{vmatrix} = 0 \quad (I)$$

Case i) Equilibrium point $(0,0,0)$:

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 2 \\ 2 & 1 & -\lambda \end{vmatrix} = 0$$

[from I putting $x=0$, $y=0$, $z=0$]

$$\Rightarrow (1-\lambda)\{(-1-\lambda)(-\lambda) - 2\times 1\} + 1.\{2\times 2 - 1\times(-\lambda)\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda+\lambda^2-2) + (4+\lambda) = 0$$

$$\Rightarrow \lambda - \lambda^2 + \lambda^2 - \lambda^3 - 2 + 2\lambda + 4 + \lambda = 0$$

$\Rightarrow \quad 2+4\lambda-\lambda^3=0$

$\Rightarrow -2-4\lambda+\lambda^3=0.$

Here, $p_0=-2,\ p_1=-4,\ p_2=0,\ p_3=1.$

According to Routh - Hurwitz criteria the equilibrium point $(0,0,0)$ will be stable if

i) $p_0>0$, ii) $p_1>0$, iii) $p_1p_2-p_0p_3>0$

But, $p_0=-2<0.$

Hence, $(0,0,0)$ is an unstable equilibrium point

Case ii) Equilibrium point $(0,0,2)$:

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -1-\lambda & -2 \\ 2 & 1 & -\lambda \end{vmatrix} = 0$$

[from I putting $x=0$, $y=0$, $z=2$]

$\Rightarrow (1-\lambda)\{(-1-\lambda)(-\lambda)-(-2)\times 1\}+1.\{(-2)\times 2-1\times(-\lambda)\}$
$=0$

$\Rightarrow (1-\lambda)(\lambda+\lambda^2+2)+(\lambda-4)=0.$

$\Rightarrow \lambda - \lambda^2 + \lambda^2 - \lambda^3 + 2 - 2\lambda + \lambda - 4 = 0$

$\Rightarrow -2 - \lambda^3 = 0$

$\Rightarrow 2 + \lambda^3 = 0.$

Here, $p_0 = 2$, $p_1 = 0$, $p_2 = 0$, $p_3 = 1$.

According to Routh-Hurwitz criteria the equilibrium point $(0,0,2)$ will be stable if

i) $p_0 > 0$; ii) $p_1 > 0$; iii) $p_1 p_2 - p_0 p_3 > 0$

Here, $p_0 = 2 > 0$ is satisfied. But, $p_1 = 0$. So $p_1 > 0$, is not satisfied. Hence $(0,0,2)$ is an unstable equilibrium point.

Case iii) Equilibrium point $(0,1,1)$;

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & -1-\lambda & 0 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$$

[from II putting $x=0$, $y=1$, $z=1$]

$\Rightarrow (1-\lambda)\{(-1-\lambda).(-\lambda)\} + (-1).\{-1.(-\lambda)\} = 0$

$\Rightarrow (1-\lambda)(\lambda+\lambda^2) - \lambda = 0$

$\Rightarrow \lambda + \lambda^2 - \lambda^2 - \lambda^3 - \lambda = 0$

$\Rightarrow -\lambda^3 = 0 \Rightarrow \lambda^3 = 0.$

Here, $p_0 = 0$, $p_1 = 0$, $p_2 = 0$, $p_3 = 1$.

According to Routh-Hurwitz criteria the equilibrium point $(0, 1, 1)$ will be stable if

i) $p_0 > 0$; ii) $p_1 > 0$; iii) $p_1 p_2 - p_0 p_3 > 0$

Here, $p_0 = 0$. Hence Routh-Hurwitz criteria not satisfied, So, $(0, 1, 1)$ is an unstable equilibrium point.

Hopf Bifurcation in an Autonomous Flow

For three dimensional flows:

We consider the following three dimensional flows:

$$\left.\begin{aligned}\frac{dx}{dt} &= f_1(x,y,z)\\ \frac{dy}{dt} &= f_2(x,y,z)\\ \frac{dz}{dt} &= f_3(x,y,z)\end{aligned}\right\} \quad (1)$$

Let (x^*, y^*, z^*) be an equilibrium point of the above flows. Then we must have

$$\left.\begin{aligned}\frac{dx}{dt}\Big|_{(x^*,y^*,z^*)} &= f_1(x^*,y^*,z^*) = 0\\ \frac{dy}{dt}\Big|_{(x^*,y^*,z^*)} &= f_2(x^*,y^*,z^*) = 0\\ \frac{dz}{dt}\Big|_{(x^*,y^*,z^*)} &= f_3(x^*,y^*,z^*) = 0\end{aligned}\right\} \quad (2)$$

Now for the equilibrium point (x^*, y^*, z^*) we arrive at the following characteristic equation:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} - \lambda & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} - \lambda & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} - \lambda \end{vmatrix} = 0 \quad (x^*, y^*, z^*)$$

$\Rightarrow p_0 + p_1\lambda + p_2\lambda^2 + p_3\lambda^3 = 0$ (say)

[p_3 is preferrably 1]

We consider the following matrix:

$$L_3 = \begin{bmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{bmatrix}$$

Hopf Bifurcation will be observed at the equilibrium point (x^*, y^*, z^*) if we have

i) $p_0 > 0$, ii) $D_1 = p_1 > 0$, iii) $D_2 = \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix}$

$= 0 \Rightarrow p_1 p_2 - p_0 p_3 = 0$ and $\frac{\partial D_3}{\partial \mu_i} \neq 0$; $i = 1, 2, 3, \ldots, n$

where μ_i's $(i = 1, 2, 3, \ldots, n)$ are the parameters involved in the flow and $D_3 = \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{vmatrix}$.

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For four dimensional flow:

We consider the following four dimensional flow:

$$\frac{dx}{dt} = f_1(x,y,z,w)$$
$$\frac{dy}{dt} = f_2(x,y,z,w)$$
$$\frac{dz}{dt} = f_3(x,y,z,w)$$
$$\frac{dw}{dt} = f_4(x,y,z,w) \qquad (1)$$

Let (x^*, y^*, z^*, w^*) be an equilibrium point of the above flow. Then we must have

$$\left.\frac{dx}{dt}\right|_{(x^*,y^*,z^*,w^*)} = f_1(x^*,y^*,z^*,w^*) = 0$$
$$\left.\frac{dy}{dt}\right|_{(x^*,y^*,z^*,w^*)} = f_2(x^*,y^*,z^*,w^*) = 0$$
$$\left.\frac{dz}{dt}\right|_{(x^*,y^*,z^*,w^*)} = f_3(x^*,y^*,z^*,w^*) = 0$$
$$\left.\frac{dw}{dt}\right|_{(x^*,y^*,z^*,w^*)} = f_4(x^*,y^*,z^*,w^*) = 0 \qquad (2)$$

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$\Rightarrow p_0 + p_1\lambda + p_2\lambda^2 + p_3\lambda^3 + p_4\lambda^4 = 0$ (say)

[p_4 is preferrably 1]

We consider the following matrix:

$$L_4 = \begin{bmatrix} p_1 & p_0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & 0 & p_4 \end{bmatrix}$$

Hopf Bifurcation will be observed at the equilibrium point (x^*, y^*, z^*, w^*) if we have

i) $p_0 > 0$; ii) $D_1 = p_1 > 0$; iii) $D_2 = \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix}$

$= p_1 p_2 - p_0 p_3 > 0$ and iv) $D_3 = \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 \end{vmatrix}$

$= p_1(p_2 p_3 - p_1 p_4) + p_0(p_1 \cdot 0 - p_3 p_3) = 0$

$\Rightarrow p_1 p_2 p_3 - p_1^2 p_4 - p_0 p_3^2 = 0$

$\Rightarrow (p_1 p_2 - p_0 p_3) p_3 = p_1^2 p_4$ as well as

$\frac{\partial D_4}{\partial \mu_i} \neq 0$; where μ_i's $(i=1,2,..,n)$ are the

parameters involved in the flow and

$$D_4 = \begin{vmatrix} p_1 & p_0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & 0 & p_4 \end{vmatrix}.$$

Problem: Discuss the phenomenon of Hopf Bifurcation for the following flow:

$$\frac{dx}{dt} = \alpha x + \beta y$$

$$\frac{dy}{dt} = x - z$$

$$\frac{dz}{dt} = -x + y \quad ; \quad x, y, z \in \mathbb{R} \; ; \; \alpha, \beta \in \mathbb{R}.$$

Ans: Given three dimensional flow:

$$\frac{dx}{dt} = \alpha x + \beta y$$

$$\frac{dy}{dt} = x - z$$

$$\frac{dz}{dt} = -x + y \quad ; \quad x, y, z \in \mathbb{R} \; ; \; \alpha, \beta \in \mathbb{R}.$$

Now, for equilibrium point we must have,

$$\frac{dx}{dt} = 0 \Rightarrow \alpha x + \beta y = 0 \rightarrow (1)$$

$$\frac{dy}{dt} = 0 \Rightarrow x - z = 0 \rightarrow (2)$$

$$\frac{dz}{dt} = 0 \Rightarrow -x + y = 0 \rightarrow (3)$$

From (2) and (3) we get, $x = y = z$.

From (1) $\alpha x + \beta x = 0 \Rightarrow (\alpha + \beta) x = 0$

$\Rightarrow x = 0$ or, $\alpha + \beta = 0$.

Here if $x = 0$, then $x = y = z = 0$. irrespective of any α, β

If $\alpha + \beta = 0$ then $x = y = z = k$. where k is any real number.

Now, here $f_1(x, y, z) = \alpha x + \beta y$

$$\therefore \frac{\partial f_1}{\partial x} = \alpha; \quad \frac{\partial f_1}{\partial y} = \beta; \quad \frac{\partial f_1}{\partial z} = 0$$

Here, $f_2(x, y, z) = x - z$.

$$\therefore \frac{\partial f_2}{\partial x} = 1; \quad \frac{\partial f_2}{\partial y} = 0; \quad \frac{\partial f_2}{\partial z} = -1.$$

Here, $f_3(x, y, z) = -x + y$

$$\therefore \frac{\partial f_3}{\partial x} = -1; \quad \frac{\partial f_3}{\partial y} = 1; \quad \frac{\partial f_3}{\partial z} = 0.$$

So, the general characteristic equation will be

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} - \lambda & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} - \lambda & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \alpha - \lambda & \beta & 0 \\ 1 & 0 - \lambda & -1 \\ -1 & 1 & 0 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \alpha - \lambda & \beta & 0 \\ 1 & -\lambda & -1 \\ -1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\alpha - \lambda)\{(-\lambda).(-\lambda) - (-1).1\}$$
$$+ \beta\{(-1).(-1) - 1.(-\lambda)\} = 0$$

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$$\Rightarrow (\alpha-\lambda)(\lambda^2+1)+\beta(1+\lambda)=0$$

$$\Rightarrow \alpha\lambda^2+\alpha-\lambda^3-\lambda+\beta+\beta\lambda=0$$

$$\Rightarrow (\alpha+\beta)+\alpha\lambda^2+(\beta-1)\lambda-\lambda^3=0$$

$$\Rightarrow -(\alpha+\beta)-\alpha\lambda^2+(1-\beta)\lambda+\lambda^3=0.$$

The above characteristic equation holds for all choices of equilibrium points as it does not contain any term of x, y or z.

Here, $p_0 = -(\alpha+\beta)$

$p_1 = -\alpha$

$p_2 = 1-\beta$

$p_3 = 1.$

Now, we take the matrix

$$L_3 = \begin{bmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{bmatrix}.$$

Case i) Equilibrium point (0,0,0):

For stability according to Routh-Hurwitz criteria we must have, i) $P_0 > 0 \Rightarrow -(\alpha+\beta) > 0$

$\Rightarrow \alpha+\beta < 0 \rightarrow (1)$

ii) $P_1 > 0 \Rightarrow -\alpha > 0$

$\Rightarrow \alpha < 0 \rightarrow (2)$

iii) $D_2 = P_1 P_2 - P_0 P_3 = 0$

$\Rightarrow (-\alpha)(1-\beta) - \{-(\alpha+\beta)\} \times 1 = 0$

$\Rightarrow -\alpha + \alpha\beta + \alpha + \beta = 0$

$\Rightarrow \beta(\alpha+1) = 0$

$\Rightarrow$ either $\beta = 0$ or $\alpha = -1$. $\rightarrow (3)$

If $\beta = 0$, then (2) and (#) are satisfied.

If $\alpha = -1$, (2) is satisfied and from (1) we get, $-1+\beta < 0 \Rightarrow \beta < 1$.

So, we can have two possibilities : (##)

(1) $\alpha < 0$, $\beta = 0$

(2) $\alpha = -1$, $\beta < 1$.

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But we also have $\frac{\partial D_3}{\partial \alpha} \neq 0$

$\frac{\partial D_3}{\partial \beta} \neq 0.$

Now, $D_3 = \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & 0 & p_3 \end{vmatrix}$

$= p_1(p_2 p_3 - p_1 \cdot 0) + p_0(p_1 \cdot 0 - p_3 \cdot p_3)$

$= p_1 p_2 p_3 - p_0 p_3^2$

$= -\alpha \cdot (1-\beta) \cdot 1 - \{-(\alpha+\beta)\} \cdot 1^2$

$= -\alpha + \alpha\beta + \alpha + \beta = \alpha\beta + \beta.$

Now, $\frac{\partial D_3}{\partial \alpha} = \beta$; $\frac{\partial D_3}{\partial \beta} = \alpha + 1.$

For Hopf Bifurcation we must have

$\frac{\partial D_3}{\partial \alpha} \neq 0 \Rightarrow \beta \neq 0$ and $\frac{\partial D_3}{\partial \beta} \neq 0$

$\Rightarrow \alpha + 1 \neq 0 \Rightarrow \alpha \neq -1.$

So, the possibilities what were arrived earlier

(1) $\alpha < 0$, $\beta = 0$

(2) $\alpha = -1$, $\beta < 1$

are not contradicting here.

So, here in the present case Hopf Bfurcation will not occur.

Saddle Node Bifurcation

Given two dimensional autonomous flow:

$$\frac{dx}{dt} = \mu - x^2$$

$$\frac{dy}{dt} = -y \quad ; \mu \in \mathbb{R}$$

For equilibrium point we must have

$$\left.\begin{aligned} \frac{dx}{dt} = 0 &\Rightarrow \mu - x^2 = 0 \Rightarrow x^2 = \mu \\ \frac{dy}{dt} = 0 &\Rightarrow -y = 0 \Rightarrow y = 0. \end{aligned}\right\} (I)$$

Case i): $\mu = 0$;

In this case, for equilibrium point we have, from (I), $\left.\begin{aligned} x^2 = 0 \\ y = 0 \end{aligned}\right\} \Rightarrow x = 0, y = 0.$

Hence $(0,0)$ is the only equilibrium point in this case.

Now, $f(x,y) = \mu - x^2 = -x^2 \ (as\ \mu = 0)$

$g(x,y) = -y.$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= -2x\,; & \frac{\partial f}{\partial y} &= 0 \\ \frac{\partial g}{\partial x} &= 0\,; & \frac{\partial g}{\partial y} &= -1. \end{aligned}\right\}$$

Now, the characteristic equation

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix}_{(0,0)} = 0$$

$$\Rightarrow \begin{vmatrix} -2x - \lambda & 0 \\ 0 & -1-\lambda \end{vmatrix}_{(0,0)} = 0$$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda(-1-\lambda) = 0 \Rightarrow \lambda = 0, -1.$

Here, $(0,0)$ is a pseudo-stable node.

Case ii): $\mu > 0$;

In this case, for equilibrium point we have

from (I) $\left.\begin{matrix} x^2 = \mu \\ y = 0 \end{matrix}\right\} \Rightarrow x = \pm\sqrt{\mu}\,, \; y = 0.$

Hence $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$ are the equilibrium points.

Now, $f(x,y) = \mu - x^2$

$g(x,y) = -y$

$\frac{\partial f}{\partial x} = -2x \; ; \; \frac{\partial f}{\partial y} = 0$

$\frac{\partial g}{\partial x} = 0 \; ; \; \frac{\partial g}{\partial y} = -1$

So the characteristic equation in general form:

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2x - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0$$

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For the equilibrium point $(\sqrt{\mu}, 0)$ the characteristic equation will be

$$\begin{vmatrix} -2\sqrt{\mu}-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2\sqrt{\mu}-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda = -1, -2\sqrt{\mu}$$

Hence $(\sqrt{\mu}, 0)$ is a stable node.

Again for the equilibrium point $(-\sqrt{\mu}, 0)$ the characteristic equation will be

$$\begin{vmatrix} 2\sqrt{\mu}-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2\sqrt{\mu}-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda = -1, 2\sqrt{\mu}.$$

Hence $(-\sqrt{\mu}, 0)$ is a saddle node.

Case iii) : $\mu < 0$:

In this case for equilibrium point we have from (I) $x^2 = \mu$ But as $\mu < 0$, $x^2 \neq \mu$.

$-y = 0$

Hence, in this case, we don't have any equilibrium point.

The entire result is tabulated below :

Cases	Equilibrium point(s)	Status
i) $\mu = 0$	$(0,0)$	Pseudo-stable node
ii) $\mu > 0$	a) $(\sqrt{\mu}, 0)$	Stable node
	b) $(-\sqrt{\mu}, 0)$	Saddle node
iii) $\mu < 0$	No equilibrium point	—

The above phenomena indicate a saddle node bifurcation.

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Transcritical Bifurcation

Given two dimensional autonomous flow:

$$\frac{dx}{dt} = -\mu x + x^2$$
$$\frac{dy}{dt} = -y \quad ; \mu \in \mathbb{R}.$$

For equilibrium point we must have

$$\left.\begin{aligned} \frac{dx}{dt} = 0 &\Rightarrow -\mu x + x^2 = 0 \\ \frac{dy}{dt} = -y &\Rightarrow -y = 0 \end{aligned}\right\} \quad (I)$$

~~$(0,0)$ and $(\mu, 0)$ are the equilibrium points.~~

Case 1): $\mu = 0$:

From (I) we get, for equilibrium point $x^2 = 0$
$y = 0$

$\Rightarrow x = 0, y = 0$

Hence $(0,0)$ is the only equilibrium point.

Again, $f(x,y) = -\mu x + x^2 = x^2$ (as $\mu = 0$)
$g(x,y) = -y.$

So, $\frac{\partial f}{\partial x} = 2x$; $\frac{\partial f}{\partial y} = 0$
$\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = -1.$

Here the characteristic equation:

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix}_{(0,0)} = 0$$

$$\Rightarrow \begin{vmatrix} 2x - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix}_{(0,0)} = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-1-\lambda) = 0 \Rightarrow \lambda = 0, -1$$

Hence $(0,0)$ is a pseudo-stable node.

Case ii): $\mu > 0$;

From (I) we get for equilibrium point, $-\mu x + x^2 = 0$

$-y = 0$

$\Rightarrow x = 0, \mu$

$y = 0$

$\therefore$ $(0,0)$ and $(\mu, 0)$ are the equilibrium points.

Here $f(x,y) = -\mu x + x^2$

$g(x,y) = -y$.

$\therefore \frac{\partial f}{\partial x} = -\mu + 2x$; $\frac{\partial f}{\partial y} = 0$

$\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = -1$.

Here the characteristic equation in general

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\mu + 2x - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0$$

Case ii: Equilibrium point $(0,0)$:

For the equilibrium point $(0,0)$ we have the characteristic equation as

$$\begin{vmatrix} -\mu - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0 \Rightarrow (-\mu - \lambda)(-1 - \lambda) = 0$$

$\Rightarrow \lambda = -1, -\mu. (\mu > 0)$

Here both the eigen values are negative.

Hence $(0,0)$ is a stable node.

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Case ii b): Equilibrium point $(\mu,0)$:

For the equilibrium point $(\mu,0)$ we have the characteristic equation as

$$\begin{vmatrix} -\mu+2\mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\mu-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda = -1, \mu\ (>0).$$

Here one root is -ve and the other is +ve. Hence, $(\mu,0)$ is a saddle node.

Case iii): $\mu<0$:

From (I) we get for equilibrium point,

$$-\mu x + x^2 = 0$$

$$-y = 0$$

$$\Rightarrow \left.\begin{matrix} x = 0, \mu \\ y = 0 \end{matrix}\right\}$$

$\therefore$ $(0,0)$ and $(\mu,0)$ are the equilibrium points.

Here $f(x,y) = -\mu x + x^2$

$g(x,y) = -y.$

$\therefore \frac{\partial f}{\partial x} = -\mu + 2x \; ; \; \frac{\partial f}{\partial y} = 0$

$\frac{\partial g}{\partial x} = 0 \; ; \; \frac{\partial g}{\partial y} = -1.$

Here the characteristic equation in general,

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\mu + 2x - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0$$

Case i a; Equilibrium point $(0,0)$:

For the equilibrium point $(0,0)$ we have the characteristic equation as

$$\begin{vmatrix} -\mu - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0 \Rightarrow (-\mu - \lambda)(-1 - \lambda) = 0$$

$\Rightarrow \lambda = -1, -\mu. \; (\mu < 0)$

Here one root is -ve and the other is +ve.

So, $(0,0)$ is a saddle node.

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Case iii b: Equilibrium Point $(\mu, 0)$:

For the equilibrium point $(\mu, 0)$ we have the characteristic equation as

$$\begin{vmatrix} -\mu + 2\mu - \lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (\mu-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda = -1, \mu \ (<0)$$

Here both the roots are -ve. Hence $(\mu, 0)$ is a stable node.

The entire result is tabulated below:

Cases	Equilibrium Point(s)	Status
i) $\mu = 0$	$(0,0)$	Pseudo-stable node
ii) $\mu > 0$	a) $(0,0)$	Stable node
	b) $(\mu, 0)$	Saddle node
iii) $\mu < 0$	a) $(0,0)$	Saddle node
	b) $(\mu, 0)$	Stable node

From the above table it is clear that as μ changes from -ve to +ve the equilibrium points $(0,0)$ as well as $(\mu, 0)$ change their characteristics. This phenomena indicate a transcritical bifurcation.

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Pitchfork Bifurcation

Given two dimensional autonomous flow:

$$\frac{dx}{dt} = \mu x - x^3$$
$$\frac{dy}{dt} = -y \quad ; \mu \in \mathbb{R}.$$

For equilibrium point we must have,

$$\left.\begin{aligned} \frac{dx}{dt} = 0 &\Rightarrow \mu x - x^3 = 0 \\ \frac{dy}{dt} = 0 &\Rightarrow -y = 0 \end{aligned}\right\} \quad (I)$$

Case i): $\mu = 0$:

For equilibrium point, from (I), $\left.\begin{aligned} -x^3 = 0 \\ -y = 0 \end{aligned}\right\} \Rightarrow x = 0, y = 0$

Here $(0,0)$ is the only equilibrium point.

Here also, $f(x,y) = \mu x - x^3 = -x^3$ (as $\mu = 0$).

$g(x,y) = -y$.

So, $\frac{\partial f}{\partial x} = -3x^2$; $\frac{\partial f}{\partial y} = 0$; $\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = -1$.

The characteristic equation:

$$\left.\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix}\right|_{(0,0)} = 0$$

$$\Rightarrow \left.\begin{vmatrix} -3x^2 - \lambda & 0 \\ 0 & -1-\lambda \end{vmatrix}\right|_{(0,0)} = 0 \Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-1-\lambda) = 0 \Rightarrow \lambda = 0, -1.$$

Hence, $(0,0)$ is a pseudo-stable node.

For $\mu \neq 0$, from (I), for equilibrium point we must have, $\mu x - x^3 = 0 \Rightarrow x = 0$

$y = 0$ (II)

Case (i) $\mu > 0$:

For equilibrium points from (I) we get, $\mu x - x^3 = 0$ (I)
$-y = 0$

$\Rightarrow x(\mu - x^2) = 0 \Rightarrow x = 0, \pm\sqrt{\mu}$ ($\because \mu > 0$)
$y = 0$ $y = 0$

So, we get here three equilibrium points viz.

$(0,0)$; $(\sqrt{\mu}, 0)$; $(-\sqrt{\mu}, 0)$.

Here $f(x,y) = \mu x - x^3$

$g(x,y) = -y$.

$\therefore \frac{\partial f}{\partial x} = \mu - 3x^2$; $\frac{\partial f}{\partial y} = 0$; $\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = -1$.

So, the characteristic equation in general

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \mu - 3x^2 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0$$

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Case ii(a): Equilibrium Point $(0,0)$:

Here the characteristic equation is

$$\begin{vmatrix} \mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\mu-\lambda)(-1-\lambda)=0 \Rightarrow \lambda = -1, \mu\ (>0)$.

Here one root is +ve and the other is -ve. Hence $(0,0)$ is a saddle node.

Case ii(b): Equilibrium Points $(\pm\sqrt{\mu}, 0)$:

Here the characteristic equation is

$$\begin{vmatrix} \mu-3\mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2\mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (-2\mu-\lambda)(-1-\lambda)=0 \Rightarrow \lambda = -1, -2\mu\ (\mu>0)$

Here both the roots are -ve. Hence $(\sqrt{\mu},0)$ and $(-\sqrt{\mu},0)$ are both stable nodes.

Case iii) $\mu < 0$:

For equilibrium points, from (I) we get,

$$\left.\begin{aligned} \mu x - x^3 &= 0 \\ -y &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x(\mu - x^2) &= 0 \\ y &= 0 \end{aligned}$$

$\Rightarrow x = 0, y = 0$ $(\because \mu < 0,\ \mu - x^2 \neq 0)$.

Here we have only equilibrium point as $(0,0)$.

Here $f(x,y) = \mu x - x^3$.

$g(x,y) = -y$

$\therefore \frac{\partial f}{\partial x} = \mu - 3x^2$; $\frac{\partial f}{\partial y} = 0$; $\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = -1$.

So, the characteristic equation, in general:

$$\begin{vmatrix} \frac{\partial f}{\partial x} - \lambda & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \mu - 3x^2 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0.$$

For the equilibrium point $(0,0)$ the characteristic equation will be

$$\begin{vmatrix} \mu-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\mu-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda = -1, \mu \ (\mu<0)$.

Here both the roots are -ve. So, we have $(0,0)$ as a stable node.

The entire result is tabulated below;

Case	Equilibrium point (s)	Status
i) $\mu = 0$	$(0,0)$	Pseudo-stable node
ii) $\mu > 0$	a) $(0,0)$	Saddle node
	b) $(\sqrt{\mu}, 0)$	Stable node
	c) $(-\sqrt{\mu}, 0)$	Stable node
iii) $\mu < 0$	$(0,0)$	Stable node.

The equilibrium point $(0,0)$ changes its characteristic as μ passes from -ve to +ve. This phenomenon indicate Pitchfork Bifurcation.

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Derivation of Logistic Map Equation from Population Growth Model

We have the standard equation of population growth as

$$\frac{dx}{dt} = (a - bx)x + k$$

Its discrete version is

$$\frac{x_{t+1} - x_t}{(t+1) - t} = (a - bx_t)x_t + k = ax_t - bx_t^2 + k$$

$$\Rightarrow x_{t+1} - x_t = ax_t - bx_t^2 + k$$

$$\Rightarrow x_{t+1} = (a+1)x_t - bx_t^2 + k \longrightarrow (1)$$

Next we consider $x_t = X_t + h$

i.e. $x_{t+1} = X_{t+1} + h$ also.

So, from (1) we get,

$$X_{t+1} + h = (a+1)(X_t + h) - b(X_t + h)^2 + k$$

$$\Rightarrow X_{t+1} + h = (a+1)X_t + ah + h - bX_t^2 - 2bhX_t - bh^2 + k$$

$$\Rightarrow X_{t+1} = (a+1-2bh)X_t - bX_t^2 + (ah - bh^2 + k) \longrightarrow (2)$$

We choose h in such a way so that

$$ah - bh^2 + k = 0 \Rightarrow bh^2 - ah - k = 0$$

i.e. $h = \dfrac{a \pm \sqrt{a^2 + 4kb}}{2b}$

So, if we take $h = \frac{a \pm \sqrt{a^2+4kb}}{2b}$ then

from (2) we get,

$$X_{t+1} = \left[a+1-2b.\left(\frac{a \pm \sqrt{a^2+4kb}}{2b}\right)\right]X_t - bX_t^2$$

$$= \left(a+1-a \mp \sqrt{a^2+4kb}\right)X_t - bX_t^2$$

$$X_{t+1} = \left(1 \mp \sqrt{a^2+4kb}\right)X_t - bX_t^2 \longrightarrow (3)$$

Now we put

$$\frac{b}{1 \pm \sqrt{a^2+4kb}} X_t = y_t \longrightarrow (4)$$

i.e.

from (3), $X_{t+1} = (1 \mp \sqrt{a^2+4kb})X_t\left[1 - \frac{b}{1 \mp \sqrt{a^2+4kb}} X_t\right]$

$$\Rightarrow \frac{1}{1 \pm \sqrt{a^2+4kb}} X_{t+1} = X_t\left[1 - \frac{b}{1 \pm \sqrt{a^2+4kb}} X_t\right]$$

$$\Rightarrow \frac{y_{t+1}}{b} = \left(\frac{1 \pm \sqrt{a^2+4kb}}{b}\right) y_t\left[1 - y_t\right]$$

$$\Rightarrow y_{t+1} = (1 \pm \sqrt{a^2+4kb})\, y_t\,(1-y_t) \longrightarrow (5)$$

Finally we put $1 \pm \sqrt{a^2+4kb} = m$

$\therefore$ we get $\boxed{y_{t+1} = m\, y_t (1-y_t)} \longrightarrow (6)$

The above is called the logistic map equation.

Analysis of Logistic Map Equation

We have the equation for Logistic Map as

$$y_{t+1} = m y_t (1 - y_t) \; ; \; m > 0.$$

Where, $y_t = \frac{b}{m} x_t$; $x_t = X_t + h$ and

either $h = \frac{a + \sqrt{a^2+4bk}}{2b}$, $m = 1 + \sqrt{a^2+4bk}$ or $h = \frac{a - \sqrt{a^2+4bk}}{2b}$, $m = 1 + \sqrt{a^2+4bk}$.

For one periodic fixed point we have,

$$y_{t+1} = y_t$$

i.e. $y_t = m y_t (1 - y_t)$

$\Rightarrow y_t = 0, \frac{m-1}{m}$.

(Case I) $h = \frac{a + \sqrt{a^2+4bk}}{2b}$, $m = 1 - \sqrt{a^2+4bk}$.

Then, for $y_t = 0$, $X_t = 0$.

$\therefore x_t = X_t + h = 0 + \frac{a + \sqrt{a^2+4bk}}{2b}$

$= \frac{a + \sqrt{a^2+4bk}}{2b}$.

Again, if $y_t = \frac{m-1}{m}$ then $\frac{b}{m} X_t = \frac{m-1}{m}$

$\Rightarrow X_t = \frac{m-1}{b}$.

$= \frac{1 - \sqrt{a^2+4bk} - 1}{b}$

$= -\frac{\sqrt{a^2+4bk}}{b} < 0$.

Then, $x_t = X_t + h = -\frac{\sqrt{a^2+4bk}}{b} + \frac{a + \sqrt{a^2+4bk}}{2b}$

$= \frac{a - \sqrt{a^2+4bk}}{2b}$.

This situation is possible only if $k < 0$. Otherwise, $x_t < 0$ which is impossible.

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Case II) $\boxed{h = \frac{a-\sqrt{a^2+4bk}}{2b}, \quad m = 1+\sqrt{a^2+4bk}}$

Then for $y_t = 0$, $x_t = 0$.

$\therefore x_t = X_t + h = 0 + \frac{a-\sqrt{a^2+4bk}}{2b}$

$= \frac{a-\sqrt{a^2+4bk}}{2b}$

This situation is possible only if $k<0$. Otherwise, $x_t < 0$, which is impossible.

If $y_t = \frac{m-1}{m}$, then $\frac{b}{m} X_t = \frac{m-1}{m}$

$\Rightarrow X_t = \frac{m-1}{b}$

$= \frac{1+\sqrt{a^2+4bk}-1}{b}$

$= \frac{\sqrt{a^2+4bk}}{b}$

Then, $x_t = X_t + h = \frac{\sqrt{a^2+4bk}}{b} + \frac{a-\sqrt{a^2+4bk}}{2b}$

$= \frac{a+\sqrt{a^2+4bk}}{2b}$.

So, for both the cases, the one period fixed points are $X_t = \frac{a \pm \sqrt{a^2+4bk}}{2b}$ (regardless the sign of k) and $x_t = \frac{a-\sqrt{a^2+4bk}}{2b}$ (only when $k<0$).

We prefer to take $h = \frac{a-\sqrt{a^2+4bk}}{2b}$, $m = 1+\sqrt{a^2+4bk}$ as only for this case $m > 0$

Next we consider, $g(y_t) = m y_t (1 - y_t)$.

$\therefore g'(y_t) = m - 2m y_t$.

So, $g'(y_t)\Big|_{y_t = 0} = m$

the one-period fixed point $y_t = 0$ is stable if $-1 < m < 1$

If we take, $m = 1 - \sqrt{a^2 + 4bk}$ then for stability we must have,

$$-1 < 1 - \sqrt{a^2 + 4bk} < 1$$

$$\Rightarrow -2 < -\sqrt{a^2 + 4bk} < 0$$

$$\Rightarrow 0 < \sqrt{a^2 + 4bk} < 2.$$

Again if we take, $m = 1 + \sqrt{a^2 + 4bk}$ then for stability we must have,

$$-1 < 1 + \sqrt{a^2 + 4bk} < 1$$

$$\Rightarrow -2 < \sqrt{a^2 + 4bk} < 0.$$ which is impossible.

That means, $y_t = 0$ is a stable one-period fixed point for $m = 1 - \sqrt{a^2 + 4bk}$ if $0 < \sqrt{a^2 + 4bk} < 2$; otherwise it is not stable. Again, $y_t = 0$ is always an unstable one-period fixed point for $m = 1 + \sqrt{a^2 + 4bk}$.

Again, $g'(y_t)\Big|_{y_t=\frac{m-1}{m}} = m - 2m\left(\frac{m-1}{m}\right)$

$= m - 2m + 2 = 2 - m.$

The one period fixed point $y_t = \frac{m-1}{m}$ is stable if $-1 < 2-m < 1 \Rightarrow -3 < -m < -1$

$\Rightarrow 1 < m < 3.$

If we take, $m = 1 - \sqrt{a^2+4bk}$, then for stability we must have,

$1 < 1 - \sqrt{a^2+4bk} < 3$

$\Rightarrow 0 < -\sqrt{a^2+4bk} < 2$

$\Rightarrow -2 < \sqrt{a^2+4bk} < 0$, which is impossible.

If we take, $m = 1 + \sqrt{a^2+4bk}$, then for stability we must have,

$1 < 1 + \sqrt{a^2+4bk} < 3$

$\Rightarrow 0 < \sqrt{a^2+4bk} < 2.$

That means, $y_t = \frac{m-1}{m}$ is always an unstable one-period fixed point for $m = 1 - \sqrt{a^2+4bk}$. Again, $y_t = \frac{m-1}{m}$ is a stable one-period fixed point for $0 < \sqrt{a^2+4bk} < 2$; otherwise it is unstable.

Hence, for both the cases we find that $x_t = \frac{a+\sqrt{a^2+4bk}}{2b}$ is a stable one-period fixed point for $0 < \sqrt{a^2+4bk} < 2$; otherwise it is unstable, and $x_t = \frac{a-\sqrt{a^2+4bk}}{2b}$ (where $k<0$) is always an unstable one-period fixed point.

Again, for two period fixed point we must have,

$y_{t+2} = y_t$.

Now, $y_{t+2} = m y_{t+1}(1 - y_{t+1})$

$= m[m y_t(1-y_t)][1 - \{m y_t(1-y_t)\}]$

$= m^2 y_t(1-y_t)(1 - m y_t + m y_t^2)$

Hence, for two period fixed points we must have,

$y_t = m^2 y_t(1-y_t)(1 - m y_t + m y_t^2)$

$\Rightarrow m^2 y_t(1-y_t)(1 - m y_t + m y_t^2) - y_t = 0$.

$\Rightarrow y_t[m^2(1-y_t)(1 - m y_t + m y_t^2) - 1] = 0$.

$\Rightarrow y_t\{m y_t - (m-1)\}\{m^2 y_t^2 - m(1+m) y_t + (1+m)\} = 0$.

For genuine two period fixed points we have

$m^2 y_t^2 - m(1+m) y_t + (1+m) = 0$

$\therefore y_t = \frac{m(1+m) \pm \sqrt{m^2(1+m)^2 - 4m^2(1+m)}}{2m^2}$

$= \frac{1+m \pm \sqrt{(1+m)^2 - 4(1+m)}}{2m}$

$= \frac{1+m \pm \sqrt{(1+m)(m-3)}}{2m}$

Now, the two period fixed point exists if $m > 3$.

Again, we consider,

$$m^2 y_t (1-y_t)(1-my_t+my_t^2) = h(y_t).$$

$$\therefore h'(y_t) = -4m^3y_t^3 + 6m^3y_t^2 - 2m^2(1+m)y_t + m^2$$

$$\therefore h'(y_t)\Big|_{y_t = \frac{1+m \pm \sqrt{(1+m)(m-3)}}{2m}}$$

$$= -m^2+2m+4.$$

If the two period fixed points are stable then

$$-1 < -m^2+2m+4 < 1.$$

Now, $-m^2+2m+4 > -1$

$\Rightarrow m^2-2m-4 < 1$

$\Rightarrow m^2-2m+1 < 6$

$\Rightarrow (m-1)^2 < 6 \Rightarrow -\sqrt{6} < m-1 < \sqrt{6}$

$\Rightarrow 1-\sqrt{6} < m < 1+\sqrt{6}.$

Again, $-m^2+2m+4 < 1$

$\Rightarrow m^2-2m-4 > -1$

$\Rightarrow m^2-2m-3 > 0$

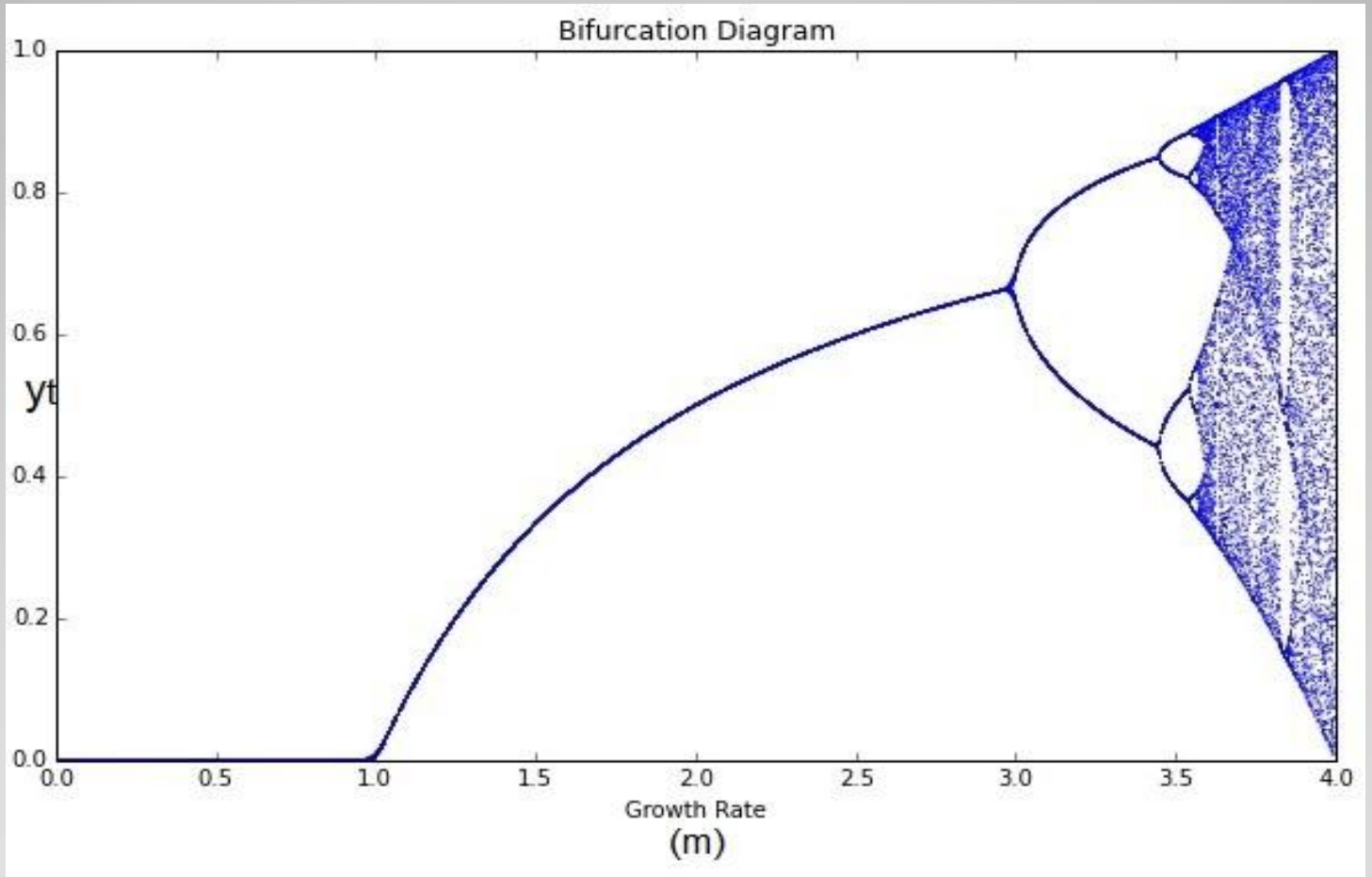
$\Rightarrow (m+1)(m-3) > 0 \Rightarrow m > 3.$

So, If $3 < m < 1+\sqrt{6}$, then the two period fixed points are stable.

Hence if $1 < m < 3$, then only one period fixed point exists and it is stable. If $3 < m < 1+\sqrt{6}$ (3.45 approximately) then both one period fixed point and two period fixed points exists, but one period fixed point is unstable and two period fixed points are stable.

With $3.45 < m < 3.54$ (approximately), the population may oscillate between four values forever. With m slightly bigger than 3.54, the population will probably oscillate between 8 values, then 16, 32, etc. The lengths of the parameter intervals which yield the same number of oscillations decrease rapidly; the ratio between the lengths of two successive such bifurcation intervals approaches the Feigenbaum constant $\delta = 4.669...$ This behaviour is an example of a period doubling cascade. At $m = 3.57$ (approximately) we can see the onset of chaos, at the end of the period-doubling cascade. We can no longer see any oscillations. Slight variations in the initial population yield drammatically different results over time, a prime characteristic of chaos. Most values beyond 3.57 exhibit chaotic behaviour, but there are still certain isolated values of m that appear to show non-chaotic behaviour; these are sometimes called islands of stability. For instance, around 3.82 there is a range of parameters m which shows oscillation between three values, and for slightly higher values of m oscillation between 6 values, then 12 etc. There are other ranges which yield oscillation between 5 values, etc; all oscillation periods do occur. Beyond $m = 4$, the values eventually leave the interval $[0, 1]$ and diverge for almost all initial values.

A Bifurcation Diagram summarizes this. The horizontal axis shows the values of the parameter m while the vertical axis shows the possible long-term values of Y.


Bifurcation Diagram
1.0
0.8
0.6
yt
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Growth Rate
(m)

References:

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2) Nonlinear Dynamics: Integrability, Chaos and Patterns-M. Lakshmanan and S. Rajasekar, Springer.

THANK YOU VERY MUCH FOR YOUR KIND ATTENTION